THE AUTOMODELLING (SIMILARITY) CASE OF HYPERSONIC FLOW PAST AN AXISYMMETRIC BODY OF A VISCOUS HEAT - CONDUCTING GAS

(AVTOMODEL'NYI SLUCHAI GIPERZVUKOVOGO OBTEKANIIA OSESIMMETRICHNOGO TELA VIAZKIM TEPLOPROVODNIM GAZOM)

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V. V. LUNEV
(Moscow)

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This paper describes an example of flow past a thin axisymmetric body of a stream with highly supersonic velocities ($M \gg 1$), when the external flow is appreciably influenced by the boundary layer.

In his paper [1] Stewartson showed for a flat plate, and in another paper [2] it was shown for the general case, that the region of flow between the surface of the body and the density discontinuity is divided into two zones, with rather clear boundaries: the viscous zone, where the boundary layer equations hold, and the nonviscous zone, the flow in which is described by the equations of motion of an ideal compressible gas, simplified on the basis of the law of plane cross-sections. It was shown also that the transport of mass through the boundary layer is negligibly small in comparison with the total transport of gas through the perturbed region, so that approximately it can be assumed that the impinging stream flows round a certain fictitious body, the surface of which coincides with the surface of the boundary layer.

Inside the boundary layer high temperatures are possible, so that in discussion of this region the thermodynamic relation is given as general a form as possible. Outside the boundary layer we can assume a perfect gas with a constant adiabatic index.

The equations of the hypersonic boundary layer on a thin axisymmetric body at the coordinates xL, yL, where x is measured from the nose along a generator of the body, whilst y is measured from the surface of the body along the normal to it, have the form [2]

$$\begin{gathered}\rho r\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-r\frac{\partial p}{\partial x}+\frac{1}{R}\frac{\partial}{\partial y}\left(r\mu\frac{\partial u}{\partial y}\right)\\ \rho r\left(u\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}\right)=ru\frac{\partial p}{\partial x}+\frac{1}{R}\frac{\partial}{\partial y}\left(r\frac{\mu}{\sigma}\frac{\partial i}{\partial y}\right)+\frac{1}{R}r\mu\left(\frac{\partial u}{\partial y}\right)^2\\ \frac{\partial(\rho ru)}{\partial x}+\frac{\partial(r\rho u)}{\partial y}=0\qquad\left(R=\frac{U_\infty\rho_\infty L}{\mu_\infty}\right)\end{gathered}\tag{1}$$

Here uU_∞, vU_∞ are the components of the velocity along the x and y axes, respectively, iU_∞^2, $p\rho_\infty U_\infty^2$, $\rho\rho_\infty$, $\mu\mu_\infty$ are the enthalpy, pressure, density and viscosity of the gas, σ is the Prandtl number, rL is the distance of the point from the axis of the body. The subscript ∞ relates to the homonymous dimensional quantity in the impinging stream. For a thin body $r = r_w(x) + y$, where $r_w(x)$ is the form of the generator.

Let us make a transformation of the variables from x, y, to x, r, and introduce subsequently Dorodnitsin's variables generalised to the axisymmetric case:

$$\xi=\frac{1}{2}\int_0^x pr_w^2dx,\qquad \eta=\int_{r_w}^r\rho r\,dr\tag{2}$$

Then Equations (1) take the form

$$\begin{gathered}u\frac{\partial u}{\partial\xi}+v_1\frac{\partial u}{\partial\eta}=-\frac{1}{\rho}\frac{dp}{d\xi}+2\frac{M^2}{R}\frac{\partial}{\partial\eta}\left(Yf\frac{\partial u}{\partial\eta}\right)\\ u\frac{\partial i}{\partial\xi}+v_1\frac{\partial i}{\partial\eta}=\frac{u}{\rho}\frac{dp}{d\xi}+2\frac{M^2}{R}\frac{\partial}{\partial\eta}\left(Y\frac{f}{\sigma}\frac{\partial i}{\partial\eta}\right)+2\frac{M^2}{R}Yf\left(\frac{\partial u}{\partial\eta}\right)^2\\ \frac{\partial u}{\partial\xi}+\frac{\partial v_1}{\partial\eta}=0\qquad\left(v_1=2\frac{u}{pr_w^2}\frac{\partial\eta}{\partial x}\bigg|_r+\frac{2r(v+ur_w')}{pr_w^2},\ Y=\frac{r^2}{r_w^2},\ f=\frac{\mu\rho}{pM^2}\right)\end{gathered}\tag{3}$$

Let us assume that the equation of state has the form $p/\rho = F(i)$. We shall assume also that f and σ are functions only of enthalpy.*

It is evident that $f \sim 1$ and $F \sim [(\kappa - 1)/\kappa]\,i$. Close to the surface of the boundary layer $r = r_\delta$ and outside this $F = [(\kappa - 1)/\kappa]\,i$. On the surface of the boundary layer $i = i_\delta \sim r_\delta^2$: inside it $i \sim 1$. Consequently, without any particular error in the final solution we can** take $i_\delta \approx 0$.

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* In actual fact, for dissociated air the functions f, F and σ depend weakly on the pressure, but this dependence will be neglected.

** Since the functions f and σ vary only slowly anyway (they are usually taken as constants), when $i_\delta \approx 0$ they can be regarded as constants, with their respective proper values.

It will be shown below that the solution of the System (3) in the case under consideration has an asymptotic character and that the quantity r_δ has therein a finite limit as $\eta \to \infty$. On this basis let us take for the System (3) the following boundary conditions*:

$$u = 1, \qquad i = 0 \quad \text{when } \eta \to \infty \tag{4}$$

$$u = v_1 = 0, \qquad i = i_w = \text{const} \qquad \text{or} \quad \frac{\partial i}{\partial \eta} = 0 \quad \text{when } \eta = 0$$

We shall seek conditions under which the solution will have the form

$$u = \varphi'(\zeta), \qquad i = i(\zeta), \qquad \zeta = \frac{\sqrt{R}\eta}{2M\sqrt{\xi}} \tag{5}$$

Substituting (5) in (3), we find that the functions $\phi(\zeta)$ and $i(\zeta)$ should in this case satisfy the conditions (4) and the system

$$(Y/\varphi'')' + \varphi\varphi'' = 2mF(i) \qquad \left(m = \frac{\xi}{p}\frac{dp}{d\xi}\right) \tag{6}$$

$$\left(Y\frac{f}{\sigma}i'\right)' + \varphi i' + Y/\varphi''^2 = -2m\varphi' F(i) \tag{7}$$

Here the function Y according to Formulas (2) has the form

$$Y = \frac{r^2}{r_w^2} = 1 + kJ(\zeta) \qquad \left(k = \frac{4M\sqrt{\xi}}{\sqrt{R}r_w^2 p}, \; J(\zeta) = \int_0^\zeta F(i)\,d\zeta\right) \tag{8}$$

Analysing Equations (6)-(7), it is easily seen that the functions ϕ'' and i decrease as $\zeta \to \infty$ not more slowly than $\exp(-h\zeta^2)$, where the constant h is equal to the value of the quantity $\sigma/2Yf$ at the surface of the boundary layer. Consequently, the integral $J(\zeta)$ in Formula (8) converges rather quickly, and the use of the above conditions for a finite value of r_δ as $\zeta \to \infty$ is indeed justified.

The solution in the form (5) will exist, evidently, in this case, if the following conditions are fulfilled:

$$m = \frac{\xi}{p}\frac{dp}{d\xi} = \text{const}, \qquad k_0 = \frac{\sqrt{\xi}}{r_w^2 p} = \text{const} \tag{9}$$

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* The condition $i_\delta \approx 0$ was used by Stewartson in his solution of the problem of an infinite plate in a gaseous space suddenly set in motion with a high supersonic velocity [3]. He showed that the use of this condition does not involve an essential error in the determination of the parameters in the bottom portion of the boundary layer and the pressure on the plate.

Expanding the relations (9) and (2), we obtain (a and b are constants)

$$r_w = ax^n, \quad r_\delta = \alpha a x^n, \qquad p = bx^{1-2n} \qquad \left(n = \tfrac{1}{2} - m, \ \alpha = \sqrt{1 + kJ(\infty)}\right) \qquad (10)$$

Accordingly, for arbitrary values of the quantities n, a and b under the assumptions which have been made, there exist automodelling (similarity) solutions of the equations of the axisymmetric boundary layer. We notice that when $k \ll 1$ the existence of automodelling solutions requires only the single condition $p = bx^{2m}$.

Let us introduce the formula for $v_0 = v + r_w' u$ – the component of velocity in the boundary layer along the axis of r. In the variables x and r, we have

$$prv_0 = -\frac{\partial}{\partial x}\int_{r_w}^{r} purdr = -\frac{2M}{\sqrt{R}}\frac{\partial}{\partial x}[\sqrt{\xi}\varphi] = -\frac{M}{2\sqrt{R}}\frac{pr_w^2}{\sqrt{\xi}}\varphi - \frac{2M}{\sqrt{R}}\sqrt{\xi}\varphi'\frac{\partial\zeta}{\partial x}$$

Determining the derivative $d\zeta/dx$ from (8), we obtain the formula

$$rv_0 = -\frac{2}{kp}F(i)\varphi + r_w r_w'(1 + kJ)\varphi'$$

Letting $\zeta \to \infty$ in this formula, we have $v_0 = r_\delta'$.

Hence, from the condition of continuity of the velocity field, it follows that in our problem the surface of the boundary layer is, as has already been mentioned above, a streamline for the external flow. In this case the solution of the equations of the nonviscous region are, as is well-known, automodelling with $r_\delta = \text{const} \times x^n$ and the pressure at the surface of the boundary layer is given by $p = cr_\delta'^2$, where c depends only on n. Comparing with (10), we have

$$n = \frac{3}{4}, \qquad b = c\alpha^2 a^2 n^2 \qquad (11)$$

For $\kappa = 1.4$ according to the paper [4] the constant $c = 0.91$. According to Newtonian theory $c = 1$.

Accordingly, for flow at highly supersonic velocities past thin axisymmetric bodies with generators $r_w = \text{const} \times x^{3/4}$ the solution of the problem of the interaction between the boundary layer and the external flow is automodelling (self-similar). We notice that in the plane case the equation of the profile leading to automodelling solutions is also $r_w = \text{const} \times x^{3/4}$.

Equations (6)-(7), in contrast to the plane case, are integro-differential equations. The parameter k entering the equations is determined according to Equations (8)-(11) by the relation

$$k\sqrt{1+kJ(\infty)} = \frac{8}{3\sqrt{c}}\frac{\chi}{M^2a^2} \qquad \left(\chi = \frac{M^3}{\sqrt{R}}\right) \tag{12}$$

Since the quantity $J(\infty)$ depends on k, then Equations (6)-(7), generally speaking, must be solved simultaneously with (12). If in Equations (6)-(7) we set f = const $\times$ $(i)^k$, σ = const and $F = [\kappa/(\kappa - 1)]\, i$, then according to (12) the solution of the problem in the case under consideration in agreement with the similarity law [2] is determined only by the parameters χ and Ma.

For small x the solution indicated above is not valid, since then the condition $u \approx 1$ is not fulfilled in the nonviscous region. Moreover, in the vicinity of the nose the shock wave is detached, and the gas behind the portion of it determined by the condition $r_\delta' \gtrsim 1$, forms in the ideal gas at the surface of the body a high entropy layer, the mass flow in which is $\psi_0 \sim \rho_\infty U_\infty L^2 (aa)^8$. In this layer [5] we have $i \approx 1$, and as long as the boundary layer, the mass flow in which $\psi \approx \rho_\infty U_\infty L^2 ka^2 a^4 x$, develops inside the high entropy layer, the automodelling solution will not hold. When $\psi \gg \psi_0$ and, consequently, when $x \gg a^6 a^4/k$, the high entropy layer will obviously not influence the characteristics of the boundary layer and the solution obtained above is valid.

Footnote (added in proof). After submission of this note to the press the author became aware of the paper [6]. In it the same problem is considered by means of the increase in thickness of the body by the thickness of the boundary layer formed (with $\sigma = 1$ and in the absence of heat conduction), which in turn is bound up with the pressure by means of the integral relations. In this paper it is deduced, strictly speaking incorrently, that there exist, besides the case $n = 3/4$, automodelling solutions when r_w = const $\times$ $x_k \ll r_\delta$, where $k > 0$ is arbitrary.

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